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BOUNDEDNESS AND OSCILLATION OF THE SOLUTIONS OF A CLASS OF OPERATOR-DIFFERENTIAL EQUATIONS

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Abstract

In the present paper sufficient conditions for boundedness of all solutions of an operator-differential equation of arbitrary order are obtained and some properties of its oscillating solutions are investigated.

1. Introduction.

In 1987 the book of Ladde, Lakshmikantham, Zhang [4] was published. In it, for the first time and sufficient details, the problems related to the oscillation and asymptotic theory of the functional differential equations were considered. Parallel to the development of the oscillation theory of the functional differential equations there began the development of the oscillation and asymptotic theory of various classes of ordinary differential equations such as differential equations with "maxima", impulsive differential equations as well as nonlinear integro-differential equations containing integral operators of Volterra type.

In The present paper by means of a single approach the oscillatory and asymptotic properties of the solutions of numerous classes of equations are investigated. Similar results were obtained in [3] and [5]. We shall also note the papers of Grace and Lalli [2], Philos [7] and Philos and Staikos [8], [9], [10].

2. Preliminary notes.

Consider the operator-differential equation

$$\begin{aligned}(1)\qquad &[r_{n+1}(t)[r_{n-2}(t)[\dots[r_1(t)\cdot x'(t)]\dots]']']'+\\ &+a(t)\cdot F((Ax)(t))=b(t)\end{aligned}$$

where $\mathscr{A}$ is an operator with certain properties.

Let $t_0 \in \mathbb{R}$ be a fixed number and $r_i \in C(t_0,\infty),n(0,\infty))$, $0<i\leq n-1$.

Introduce the following notation:

$$(L_0x)(t)=x(t)$$

$$(L_ix)(t)=r_i(t)[(L_ix)(t)]',\ 1\leq i\leq n,\ r_n(t):\equiv 1.$$

By $\mathscr{D}_n$ denote the set of all functions $x\in C([T_x,\infty);\mathbb{R})$ such that the functions L_ix $(0\leq i\leq n)$ exist and are continuous in $[T_x,\infty)$.

DEFINITION 1. *The function* $x:[T_x,\infty)\to\mathbb{R}$ *is said to be a solution of equation (1) if* $x\in\mathscr{D}_n$ *and* x *satisfies equation (1) for* $t\geq\max\{T_x,T_{Ax}\}$.

DEFINITION 2. *A given function* $u:[t_0,\infty)\to\mathbb{R}$ *is said to eventually enjoy the property* P *if there exists a point* $t_{p,u}\geq t_0$ *such that for* $t\geq t_{p,u}$ *it enjoys the property* P.

DEFINITION 3. *The solution x of equation (1) is said to be regular if* $\sup |x(t)| > 0$ *eventually.*

DEFINITION 4. *The regular solution x of equation (1) is said to oscillate if* $\sup\{t; x(t) = 0\} = \infty$. *Otherwise, the regular solution x is said to be nonoscillating.*

Introduce the following conditions:

H1. $r_i \in C([t_0, \infty), (0, \infty))$, $1 \leq i \leq n - 1$

H2. $a \in C([t_0, \infty); \mathbb{R})$

H3. $b \in C([t_0, \infty); \mathbb{R})$

H4. $F \in C(\mathbb{R}, \mathbb{R})$

H5. $\mathscr{A} : \mathscr{D}_n \to C([T_{Ax}, \infty); \mathbb{R})$, $T_{Ax} \geq t_0$

3. Main results.

THEOREM 1. *Let the following conditions hold:*

1. *Conditions H1-H5 are met.*

2. $$\lim_{t\to\infty} \int_{t_0}^{t} \frac{1}{r_1(s_1)} \int_{t_0}^{s_1} \frac{1}{r_2(s_2)} \dots \int_{t_0}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} ds_{n-1} \dots ds_2 ds_1 < \infty$$

3. $\int^{\infty} |a(t)| dt < \infty$.

4. $\int^{\infty} |b(t)| dt < \infty$.

5. *$F(u)$ is a bounded function in $\mathbb{R}$.*

Then:

1. All solutions x of equation (1) are bounded.

2. All oscillating solutions x of equation (1) enjoy the property $\lim_{t\to\infty} x(t) = 0$.

Proof 1. Let x be a solution of the differential equation (1) in the interval $[T, \infty)$, $T \geq t_0$. From condition 3 and condition 4 of Theorem 1 it follows that there exist constants A_T and B_T such that

$$0 \leq \mathscr{A}_R = \int_T^{\infty} |a(t)|dt < \infty$$

$$0 \leq B_T = \int_T^{\infty} |B(t)|dt < \infty$$

Introduce the following notation:

$$R_k(t) = \int_T^t \frac{1}{r_1(s_1)} \int_T^{s_1} \frac{1}{r_2(s_2)} \cdots \int_T^{s_{k-1}} \frac{1}{r_k(s_k)} ds_k \ldots ds_2 ds_1 \tag{2}$$

$$\mathscr{M} = \sup |F(u)|, u \in \mathbb{R}. \tag{3}$$

From condition 5 of Theorem 1 it follows that $\mathscr{M}$ is a finite number. From condition 2 of Theorem 1 we obtain that

$$0 \leq S_k = \lim_{t\to\infty} R_k(t) < \infty, \ 1 \leq k \leq n-1.$$

From Taylor's generalized formula it follows that for $t \geq T$.

$$x(t) = x(T) + \sum_{k=1}^{n-1} (L_k x)(T) \cdot R_k(t) +$$
$$+ \int_T^t \frac{1}{r_1(s_1)} \int_T^{s_1} \frac{1}{r_2(s_2)} \cdots$$
$$\cdots \int_T^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \int_T^{s_{n-1}} (L_n x)(s) ds ds_{n-1} \ldots ds_2 ds_1$$

$$|x(t)| \leq |x(T)| + \sum_{k=1}^{n-1} |(L_k x)(T)| \cdot R_k(t) +$$

$$+ R_{n-1}(t) \int_T^t |(L_n x)(s)| ds \leq$$

$$\leq |x(T)| \sum_{k=1}^{n-1} |(L_k x)(T)| R_k(t) + R_{n-1}(t) \int_T^t |b(s)| ds +$$

$$+ R_{n-1}(t) \int_T^t |a(s)| \cdot |F((\mathscr{A} x)(s))| ds \leq$$

$$\leq |x(T)| + \sum_{k=1}^{n-1} |(L_k x)(T)| s_k + s_{n-1} B_T + \mathscr{M} s_{n-1} A_T < \infty.$$

i.e. $|x(t)| < \infty$.

Proof 2. Let x be an oscillating solution of equation (1) in the interval $[T_0, \infty)$, $T_0 \geq t_0$. From conditions 2, 3 and 4 of Theorem 1 it follows that we can choose $\varepsilon > 0$ and $T \geq T_0$ so that

$$\int_T^\infty |(a(t)| dt < \frac{\varepsilon}{2\mathscr{M} S_{n-1}}, \quad \int_T^\infty |b(t)| < \frac{\varepsilon}{2S_{n-1}}$$

where $S_4 n - 1$, $\mathscr{M}$ are introduced by (2) and (3).

From the fact that x is an oscillating solution of equation (1) in the interval $[T, \infty)$ it follows that $L_i x$ $(1 \leq i \leq n-1)$ are also oscillating functions in $[T, \infty)$. Let τ_i $(0 \leq i \leq n-1)$ be such that $x(\tau_i) = 0$ and $\tau_0 \geq \tau_1 \ldots \geq \tau_{n-1} T$. Then $(L_i x)(\tau_i) = 0$, $0 \leq i \leq n-1$.

Integrating equation (1) for $t \geq \tau_0$ we obtain that

$$x(t) = \int_{\tau_0}^t \frac{1}{r_1(s_1)} \int_{\tau_1}^{s_1} \frac{1}{r_2(s_2)} \ldots$$

$$\ldots \int_{\tau_{n-2}}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \int_{\tau_{n-1}}^{s_{n-1}} (L_n x)(s) ds ds_{n-1} \ldots ds_2 ds_1$$

i.e.

$$\begin{aligned}
|x(t)| \leq & \int_{\tau_0}^{t} \frac{1}{r_1(s_1)} \int_{\tau_1}^{s_1} \frac{1}{r_2(s_2)} \cdots \\
& \cdots \int_{\tau_{n-2}}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})} \int_{\tau_{n-1}}^{s_{n-1}} |(L_n x)(s) ds ds_{n-1} \ldots ds_2 ds_1 \leq \\
& \leq R_{n-1}(t) \int_T^t |(L_n x)(s)| ds \leq R_{n-1}(t) \int_T^t |b(s)| ds + \\
& + R_{n-1}(t) \int_T^t |a(s)|\, F((\mathcal{A} x)(s))| ds \leq \\
& \leq S_{n-1} \int_T^{\infty} |b(s)| ds + S_{n-1} \mathcal{M} \int_T^{\infty} |a(s)| ds < \varepsilon .
\end{aligned}$$

Hence for any $\varepsilon > 0$ and any sufficiently large $t \geq \tau_0$ we have $|x(t)| < \varepsilon$, i.e. $\lim\limits_{t \to \infty} x(t) = 0$. ■

COROLLARY 1. *Let the following conditions hold:*

1. $(\mathcal{A}(x)(t) = \max\limits_{S \in M(t)} x(s)$, *where* $\mathcal{M}(t) = [p(t), q(t)]$ *is a compact subset of the interval* $[t_0, \infty)$ *for* $t \geq t_0$ *and* $p, q \in C([t_0, \infty); \mathbb{R})$, $p(t) \leq q(t)$, $\lim\limits_{t \to \infty} p(t) = \infty$

2. Conditions H1-H4 are met.

3. Conditions 2, 3, 4 and 5 of Theorem 1 hold.

Then:

1. Each solution x *of the differential equation*

$$(L_n x)(t) + a(t) F\left(\max_{S \in / M(t)} x(s)\right) = b(t) \tag{4}$$

is bounded.

2. Each oscillating solution x *of equation (4) enjoys the property* $\lim\limits_{t \to \infty} x(t) = 0$.

Proof In order to prove Corollary 1 it is necessary to check that condition 1 of Corollary 1 implies condition H5, i.e. we must prove that $(\mathscr{A}x)(t) = \max_{S \in \mathscr{M}(t)} x(s)$ is a continuous function in the interval $[t_0, \infty)$. To this end we introduce the following metric in the set of all intervals:

$$\Im = \{[p, q], p, q \in \mathbb{R} = (-\infty, +\infty);\ p \leq q\} :$$

$$\rho([p, q], [\bar{p}, \bar{q}]) = \max\{|p - \bar{p}|, |q - \bar{q}|\}.$$

Then the mapping $P : \bar{R}_+ \to \Im \quad (t - [p(t), q(t)])$ is continuous, where $\bar{R}_+ = [0, \infty), p$ and q are continuous functions for $t \geq t_0$.

Analogously, the mapping $Q : \Im \to \mathbb{R}$ $([p, q] \to \max_{S \in [p,q]} x(s))$ is continuous. Then the sperposition of P and Q is also a continuous mapping, i.e. $(\mathscr{A}x)(t) = \max_{\in \mathscr{M}(t)} x(s) = Q(P(t))$ is a continuous function in $[t_0, \infty)$. (cf. [1]).

EXAMPLE 1. Consider the differential equation

$$[t^{-2}[t^5 x'(t)]']' + e^{\frac{1}{t}} t^{-2} \frac{1}{e^{|\max_{s \in [t,t+1]} x(s)|}} = \frac{1}{t^2}, t \geq 1. \tag{5}$$

Here the functions $r_0(t) = 1$, $r_1(t) = t^5$, $r_2(t) = t^{-2}$, $a(t) = t^{-2} e^{\frac{1}{t}}$, $b(t) = t^{-2}$, $F(u) = e^{-|u|}$n, $(\mathscr{A}x)(t) = \max_{s \in [t,t+1]} x(s)$ satisfty the conditions of Corollary 1. Then each solution x of equation (5) is bounded and for each oscillating solution x we have $\lim_{t \to \infty} x(t) = 0$. For instance, $x(t) = \dfrac{1}{t}$ is a nonoscillating bounded solution of equation (5).

COROLLARY 2. *Let the following conditions hold:*

1. $(\mathscr{A}x)(t) = x(g(t))$, *where* $g \in C([t_0, \infty); \mathbb{R})$, $\lim_{t \to \infty} g(t) = +\infty$, $g(t) \leq t$ *for* $t \geq t_0$.

2. Conditions H1-H4 hold.

3. Conditions 2, 3, 4 and 5 of Theorem 1 are satisfied.

Then:

1. Each solution x of the differential equations

$$(L_n x)(t) + a(t)F(x(g(t))) = b(t) \tag{6}$$

is bounded.

2. Each oscillating solution x of equation (6) enjoys the property $\lim_{t\to\infty} x(t) = 0$.

The proof of Corollary 2 follows immediately from Theorem 1 since from condition 1 of Corollary 2 we deduce condition H5.

EXAMPLE 2. Consider the differential equation

$$[t^{-2}[t^5 x'(t)]']' + e^{\frac{2}{t}} t^{-2} e^{-|x(\frac{t}{2})|} = t^{-2}, t \geq 1.$$

Here the functions $r_0(t) = 1$, $r_1(t) = t^5$, $r_2(t) = t^{-2}$, $F(u) = e^{-|u|}$, $a(t) = e^{\frac{2}{t}} t^{-2}$, $b(t) = t^{-2}$, $(\mathscr{A} x)(t) = x\left(\frac{t}{2}\right)$ satisfy the conditions of Corollary 2. Then each oscillating solution tends to zero as $t \to \infty$ and each solution is bounded. For instance, $x(t) = \dfrac{1}{t}$ is a nonoscillating bounded solution.

EXAMPLE 3. Consider the differential equation

$$\begin{aligned}[t^{-1}[e^t x'(t)]']' - \frac{\sin t|\sin t|}{e^{2t+\pi}} \cdot \frac{1}{|x(t-\pi)|} = \\ = e^{-t}\left[\left(\frac{2}{t} - \frac{1}{t^2}\right)\sin t + (4t^{-1} + 3t^{-2})\cos t\right] - \\ - e^{-3t}\sin t, t \geq 1.\end{aligned}$$

Here the functions $r_1(t) = e^t$, $r_2(t) = t^{-1}$, $a(t) = -\dfrac{\sin t|\sin t|}{e^{2t+\pi}}$ $F(u) = \dfrac{1}{|u|}$, $(\mathscr{A}x)(t) = x(t-\pi)$, $b(t) = e^{-t}[(2t^{-1} - t^{-2})\sin t + (4t^{-1} + 3t^{-2})\cos t] - e^{-3t}\sin t$ satisfy the conditions of Corollary 2. Then all solutions of the equations are bounded, and the oscillating ones tend to zero as $t \to \infty$. For instance, $x(t) = \dfrac{\sin t}{e^{2t}}$ is an oscillating bounded solution and $\lim\limits_{t\to\infty} x(t) = 0$.

COROLLARY 3. *Let the following conditions hold:*

1. $(\mathscr{A}x)(t) = \displaystyle\int_{t-a}^{t} k(t,s)x(s)ds$, *where* $a = const > 0$,

$$k \in C([t_0, \infty) \times [t_0 + a, \infty); \mathbb{R})$$

2. Conditions H1-H4 are met.

3. Conditions 2, 3, 4 and 5 of Theorem 1 are satisfied.

Then: 1. Each solution x the equation

$$(L_n x)(t) + a(t)F\left(\int_{t-a}^{t} k(t,s)x(s)ds\right) = b(t) \tag{7}$$

is bounded.

2. Each oscillating solution x of equation (7) enjoys the property $\lim\limits_{t\to\infty} x(t) = 0$.

Proof From $k \in C([t_0, \infty) \times [t_0 + a, \infty); \mathbb{R})$ it follows that $\mathscr{A}x \in C([t_0, \infty); \mathbb{R})$, i.e. condition H5 is met. Then from Theorem 1 there follows the validity of Corolalry 3.

EXAMPLE 4. Consider the differential equation

$$[t^3[t^2x'(t)]']' + (2\sin t \cdot t^{-3} - t^{-2}\cos t + t^{-2})\cdot$$

$$\cdot \frac{1}{e^{|u(t)|}} = \frac{1}{t_2},\ t \geq 2\pi$$

where

$$u(t) = \int_{t-2\pi}^{t} st^{-1}x(s)ds.$$

The functions $r_1(t) = t^2$, $r_2(t) = t^3$, $F(u) = e^{-|u|}$ $a(t) = 2t^{-3}\sin t - t^{-2}\cos t + t^{-2}$, $b(t) = t^{-2}$, $(\mathscr{A}x)(t) = u(t)$, $k(t,s) = st^{-1}$ satisfy the conditions of Corollary 3. Then each solution of the equation is bounded and each bounded oscillating solution tends to zero as $t \to \infty$. For instance, $x(t) = \dfrac{\sin t}{t}$ is an oscillating bounded solution for which $\lim\limits_{t\to\infty} x(t) = 0$.

Consider the class of funtional differential equations with distributed delay

$$(L_n x)(t) + a(t)F\left(\int_{\alpha(t)}^{\beta(t)} f(t, x(t+s))d_s r(t,s)\right) = \tag{9}$$
$$= b(t),\ t \geq t_0$$

where

$$(\mathscr{A}x)(t) = \int_{\alpha(t)}^{\beta(t)} f(t, x(t+s))d_s r(t,s) \tag{10}$$

DEFINITION 5. *The function $x \in C^n([t_0,\infty);\mathbb{R})$ is said to be a solution of equation (9) if it satisfies (9) for $t \geq t_0$ and $x^{(k)}(t+s) = \phi_k(t+s)$ for $t+s < t_0$, $\phi_k(t_0) = x^{(k)}(t_0+0)(0 \leq k \leq n-1)$, where $\phi_k(t)$ are given functions defined and continuous in*

$$\left[\inf_{t\to\infty}(t+\alpha(t)), t_0\right].$$

Introduce the following conditions:

H6. $f \in C([t_0,\infty)\times\mathbb{R};\mathbb{R})$

H7. $\alpha, \beta \in C([t_0,\infty);\mathbb{R}), \alpha(t) < \beta(t)$, for $t \geq t_0$

H8. $r(t, s) : [t_0, \infty) \times [\alpha(t), \beta(t)] \to \mathbb{R},\ r(t, 0) = 0$

$$r(t, \alpha(t)), r(t, \beta(t)) \in C([t_0, \infty) \times \mathbb{R}; \mathbb{R})$$

H9. For any fixed $t \geq t_0$ the function $r(t, s)$ is increasing with respect to the variable S in the interval $[\alpha(t), \beta(t)]$.

H10. For any $t_1 \geq t_0$ the following condition holds

$$\lim_{t \to t_1} \int_{\max(\alpha(t), \alpha(t_1))}^{\min(\beta(t), \beta(t_1))} |r(t, s) - r(t_1, s)| ds = 0$$

H11. $Var_{\alpha(t)}^{\beta(t)} r(t, s) \leq m(t)$, where the function $m(t)$ is locally integrable in $[t_0, \infty)$.

COROLLARY 5. *Let the following conditions hold:*

1. *Conditions H1-H4, H6-H11 are met.*
2. *Conditions 2, 3, 4 and 5 of Theorem 1 are satisfied.*

Then:

1. *Each solution of equation (9) is bounded.*
2. *Each oscillating solution of equation (9) enjoys the property* $\lim_{t \to \infty} x(t) = 0$.

Proof We shall note that if conditions H6-H11 are met, then the Stieltjes integral (10) exists and, as proved in [6], if $x \in C([t_0, \infty); \mathbb{R})$ then $\mathscr{A} x \in C([t_0, \infty); \mathbb{R})$ as well. □

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